

Solutions to Coursework #8
MAS231 Geometry II, 2008

$$\begin{aligned}
 1. a) F_I^{-1} F_{II} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \\
 &= \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \\
 &= \frac{1}{EG-F^2} \begin{pmatrix} GL-FM & GM-FN \\ -FL+EM & -FM+EN \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \text{trace}(F_I^{-1} F_{II}) &= \frac{1}{2} \left(\frac{GL-FM}{EG-F^2} + \frac{-FM+EN}{EG-F^2} \right) \\
 &= \frac{GL-2FM+EN}{2(EG-F^2)}
 \end{aligned}$$

$$b) \cancel{V_1^T F_I V_2} t_1 \cdot t_2 = (a_1 \sigma_u + b_1 \sigma_v) \cdot (a_2 \sigma_u + b_2 \sigma_v)$$

$$\begin{aligned}
 &= a_1 a_2 \sigma_u \cdot \sigma_u + a_1 b_2 \sigma_u \cdot \sigma_v + b_1 a_2 \sigma_v \cdot \sigma_u + b_1 b_2 \sigma_v \cdot \sigma_v \\
 &= (a_1 \ b_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = a_1 a_2 E + (a_1 b_2 + a_2 b_1) F + b_1 b_2 G \\
 &= (a_1 \ b_1) \begin{pmatrix} a_2 E + b_2 F \\ a_2 F + b_2 G \end{pmatrix} = a_1 (a_2 E + b_2 F) + b_1 (a_2 F + b_2 G) \\
 &= a_1 a_2 E + (a_1 b_2 + a_2 b_1) F + b_1 b_2 G
 \end{aligned}$$

$$\text{So } t_1 \cdot t_2 = V_1^T F_I V_2,$$

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2 a) $\sigma_u = (1, 0, 2u)$
 $\sigma_v = (0, 1, 3v^2)$

$$E = \sigma_u \cdot \sigma_u = 1 + 4u^2$$

$$F = \sigma_u \cdot \sigma_v = 6uv^2$$

$$G = \sigma_v \cdot \sigma_v = 1 + 9v^4$$

$$\sigma_u \times \sigma_v = (-2u, -3v^2, 1)$$

$$N_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-2u, -3v^2, 1)}{\sqrt{4u^2 + 9v^4 + 1}}$$

~~$\sigma_u = (0, 0, 2)$~~ $\sigma_{uu} = (0, 0, 2), \sigma_{uv} = (0, 0, 0), \sigma_{vv} = (0, 0, 6v)$

$$L = \sigma_{uu} \cdot N_\sigma = \frac{2}{\sqrt{4u^2 + 9v^4 + 1}}$$

$$M = \sigma_{uv} \cdot N_\sigma = 0$$

$$N = \sigma_{vv} \cdot N_\sigma = \frac{6v}{\sqrt{4u^2 + 9v^4 + 1}}$$

$$b) K_G = \frac{LN - M^2}{EG - F^2} = \frac{12v}{(4u^2 + 9v^4 + 1)[(1 + 9v^4)(1 + 4u^2) - \{6uv^2\}^2]}$$

$$= \frac{12v}{(4u^2 + 9v^4 + 1)(1 + 9v^4 + 4u^2)} = \frac{12v}{(1 + 4u^2 + 9v^4)^2}$$

Since $(1 + 4u^2 + 9v^4)^2 > 0$, $K_G > 0$ if $v > 0$, $K_G = 0$ if $v = 0$, $K_G < 0$ if $v < 0$.

(elliptic) (parabolic) (hyperbolic)

$$3.a) \sigma_u = (1, 0, \frac{u}{\sqrt{u^2+v^2}}) \quad \sigma_v = (0, 1, \frac{v}{\sqrt{u^2+v^2}})$$

$$N_\sigma = \frac{1}{\sqrt{2}} \left(-\frac{u}{\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1 \right) \text{ as before}$$

$$\sigma_{uu} = (0, 0, \frac{v^2}{(u^2+v^2)^{3/2}})$$

$$\sigma_{vv} = (0, 0, \frac{u^2}{(u^2+v^2)^{3/2}})$$

$$\sigma_{uv} = (0, 0, -\frac{uv}{(u^2+v^2)^{3/2}})$$

$$F_{II} = \frac{1}{\sqrt{2}(u^2+v^2)^{3/2}} \begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix}$$

$$b) F_I = \begin{pmatrix} 2u^2+v^2 & uv \\ uv & u^2+2v^2 \end{pmatrix} \cdot \frac{1}{u^2+v^2}$$

$$LN - M^2 = \frac{v^2 u^2 - (-uv)^2}{2(u^2+v^2)^3} = 0 \Rightarrow K_G = 0$$

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

$$\begin{aligned} EG - F^2 &= \frac{1}{(u^2+v^2)^2} \left[(2u^2+v^2)(u^2+2v^2) - (uv)^2 \right] \\ &= \frac{1}{(u^2+v^2)^2} [2u^4 + 4u^2v^2 + 2v^4] \\ &= \frac{2(u^2+v^2)^2}{(u^2+v^2)^2} = 2. \end{aligned}$$

$$H = \frac{1}{4\sqrt{2}(u^2, v^2)^{5/2}} \left[v^2(u^2 + 2v^2) - 2(-uv)(uv) + u^2(2u^2 + v^2) \right]$$

$$= \frac{1}{4\sqrt{2}(u^2 + v^2)^{5/2}} \left[2v^4 + 2u^4 + 4u^2v^2 \right]$$

$$= \frac{1}{4\sqrt{2}(u^2 + v^2)^{5/2}} 2(u^2 + v^2)^2$$

$$= \frac{1}{2\sqrt{2}\sqrt{u^2 + v^2}}$$

$$c) K_1, K_2 = H \pm \sqrt{H^2 - K_0}$$

$$= \cancel{K} H \pm |H|$$

$$= 0, \frac{1}{\sqrt{2}\sqrt{u^2 + v^2}}$$

4a) Let γ be a unit speed curve in a surface S and let t_1 and t_2 be principal unit vectors at a point of γ . Let θ be the angle that γ' makes with respect to t_1 . Then the normal curvature K_n of γ equals

$$K_n = K_1 \cos^2 \theta + K_2 \sin^2 \theta.$$

$$b) \quad F_I^{-1} F_{II} \begin{pmatrix} a \\ b \end{pmatrix} = O \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow F_{II} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left. \begin{aligned} v^2 a - uvb &= 0 \\ -uva + u^2 b &= 0 \end{aligned} \right\} \Rightarrow va = ub$$

is solved by $a=u, b=v$.

$$t_1 = \frac{u\sigma_u + b\sigma_v}{\|u\sigma_u + b\sigma_v\|}$$

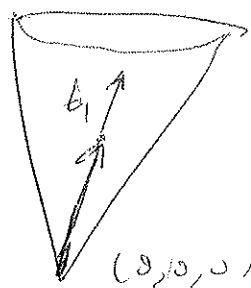
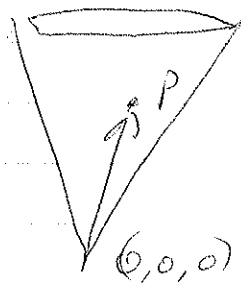
$$u\sigma_u + b\sigma_v = (u, v, \sqrt{u^2 + v^2})$$

$$\Rightarrow t_1 = \frac{(u, v, \sqrt{u^2 + v^2})}{\sqrt{2} \sqrt{u^2 + v^2}}$$

c) Take a point on S .

$$p = \sigma(u, v) = (u, v, \sqrt{u^2 + v^2})$$

t_1 is a positive multiple of p , so it points in the same direction as p



A line with tangent vector p is the line through $(0,0,0)$ and p , which is $\gamma(t) = t p$.
Its normal curvature $K_n = K_1 = 0$.